Localization and position operators in Möbius covariant theories.

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Plan of the talk

▶ **Localization**: As emerging from symmetry.
▶ The case of Mübius covariance.
▶ New aspect: **Position operators** arising from a modification of the generators of the group.
▶ **Example**: Massless KG scalars on 2D Minkowski.

**References**

Motivations

- **Causality** is an important concept in relativistic physics.

  “Spatially separated events cannot interact.”

- In QFT at level of “second quantization”. Local observables are charactered by $\mathbb{R}$-linear spaces of local wave-functions.

- It is **not** completely **intrinsic**. It seems to depend on the particular representation of the functions.

- *Brunetti Guido and Longo*: Localization ($\mathbb{R}$-linear spaces) descends from symmetrey group.

- Do observables compatible with this localization exsit?

- We analyze the case of Möbius covariant theories.
Is it a trivial task?

- **Quantum mechanics:** *Example:* Particle on the line. $L^2(\mathbb{R}, dx)$ states, $|\psi(x)|^2$ probability distribution.

- Coordinate: $X : \psi(x) \mapsto x\psi(x)$, self-adjoint operator.

- Local states in $[a, b]$ are: $L^2([a, b], dx) \subset L^2(\mathbb{R}, dx)$.

- If $\psi \in L^2([a, b], dx)$ and $\|\psi\| = 1$
  
  $$a \leq (\psi, X\psi) \leq b$$

  We say $X$ its compatible with locality.
Relativistic situation

In relativistic theories

- **Example:** Scalar KG field on 2D Minkowski.
- Chose a space-like Hypersurface, then
- *Localization* and *coordinate* can be defined as above. This is called Newton Wigner (NW) localization.

- **Problem:** NW Localization is not preserved by evolution.
  
  (Classical information cannot travel faster then light?).

It seems not Physically reasonable.
Quantization scheme and localization

For flat spacetime:

- **First quantization:** \textit{(a la Wigner)}
  
  - One-particle Hilbert space $\mathcal{H}$.
  
  - (anti)-unitary representation of the Poincarré group.

- **Second quantization:**
  
  - Consider the Fock space $\mathcal{F} := \mathcal{F}(\mathcal{H})$ built by $\mathcal{H}$ and the vacuum $\Omega$.
  
  - Weyl operators $W(\psi)$ on $\mathcal{F}$. 

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Localization: Operators need to be smeared.

\( \mathcal{O} \) a region of spacetime.
Consider real local function with support in a region \( \mathcal{O} \).
By means of causal propagator \( E \).

\[
f : \mathcal{O} \to \mathbb{R}, \quad \mathcal{K}_\mathcal{O} := \{ \psi_f \in \mathcal{H} | \psi_f = Ef, D(f) \subset \mathcal{O} \}
\]
\( \mathcal{K}_\mathcal{O} \) is a \( \mathbb{R} \)-linear subset of the one-particle Hilbert space \( \mathcal{H} \).

von Neumann algebras. \( \mathcal{A}(\mathcal{O}) := \{ W(\psi) | \psi \in \mathcal{K}_\mathcal{O} \}'' \)

if \( \mathcal{O} \) is a double cone \( \mathcal{A}(\mathcal{O}) \) is in standard form:
\( \Omega \) is cyclic and separating.
Digression Tomita Takesaki modular theory.

- Then if \( A \in \mathcal{A} \) (standard) exists an operator \( S \) from \( \mathcal{A}\Omega \) to \( \mathcal{A}\Omega \) realizing the star operation

\[
SA\Omega = A^*\Omega
\]

- Has a polar decomposition \( S := J\Delta^{1/2} \)
- \( \Delta \) self-adj. positive. \( \Delta^{it}\Delta^{-it} = A \) (modular transf.)
- \( J \) is an anti-unitary operator. \( JA\Omega = \mathcal{A}' \) (modular conj.)
- \( \mathcal{A} \) on \( \Omega \) satisfy the KMS condition w.r. to modular transf.
- For Wedges in Minkowski spacetime, have a geometrical meaning: \( J \) is a Reflection and \( \Delta^{it} \) are Boosts (Bisognano Wichmann)
- Be \( \psi = A\Omega \), with \( A^* = A \), in the one particle Hilbert state then: \( S\psi = \psi \). And also if \( \psi \in \mathcal{K} \): \( S\psi = \psi \).
New scheme

Revert the point of view:

- Recognize $J_\mathcal{O}$ and $\Delta_\mathcal{O} = e^{-D_\mathcal{O}}$ within the group of symmetry for sufficiently many local sets $\mathcal{O}$.

- Consider $S_\mathcal{O} := J_\mathcal{O} \Delta_\mathcal{O}^{1/2}$.

- Assume $\mathcal{K}_\mathcal{O} := \{\psi | S\psi = \psi\}$ as a definition for $\mathbb{R}$-linear subspace of $\mathcal{H}$ of object local in $\mathcal{O}$.

Properties:

- If $\mathcal{O}_1 \subset \mathcal{O}_2$ then $\mathcal{K}_{\mathcal{O}_1} \subset \mathcal{K}_{\mathcal{O}_2}$ \hfill (Isotony)
- If $\mathcal{O}_1$ and $\mathcal{O}_2$ spatially separated $\mathcal{K}_{\mathcal{O}_1} \cap \mathcal{K}_{\mathcal{O}_2} = \emptyset$ \hfill (Locality)
- Local function: dense in $\mathcal{H} := \overline{\mathcal{K}_\mathcal{O} + i\mathcal{K}_\mathcal{O}}$. 

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Möbius group: geometric aspects

Conformal transformations of $\mathbb{C}$ where $S^1$ is fixed.

\[ x \rightarrow \frac{ax + b}{cx + d}, \quad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in PSL(2, \mathbb{R}). \]

$PSL(2, \mathbb{R})$ transformation on $\mathbb{P}\mathbb{R}$.

$j : x \rightarrow -x$ in $\mathbb{R} \cup \{\infty\}$ involution.

Iwasawa decomposition: $g \in PSL(2, \mathbb{R})$

\[ g := T(x)\Lambda(y)P(z), \quad x, y, z \in \mathbb{R}, \]

$h, d, c$: generators

\[ [h, d] = h, \quad [c, d] = -c, \quad [c, h] = 2d. \]
**Local sets:** $I \subset \mathcal{I}$ proper interval $I = [a, b]$ in $\mathbb{R}$.

$\forall I$, the decomposition: $g : = T_I(x)\Lambda_I(y)P_I(z)$ and a $j_I$ exist.

**(A) Reflection covariance:** $j_I$ maps $I$ to $I'$ and $j_{gI} = gj_Ig^{-1}$.

**(B) $\Lambda$ covariance:** $\Lambda_I(t)$ maps $I$ to $I$ and $\Lambda_{gI}(t) = g\Lambda_I(t)g^{-1}$.

**(C) Positive inclusions:**

- If $t > 0$, $T_I(t)$ maps $I$ to $I_t \subset I$ and
  $$\Lambda_I(b)T_I(t)\Lambda_I(-b) : = T_I(e^{2\pi b}t);$$

- If $p < 0$, $P_I(p)$ maps $I$ to $I_p \subset I$ and
  $$\Lambda_I(b)P_I(p)\Lambda(-b) : = P_I(e^{-2\pi b}p).$$

**Lesson:** A particular decomposition selects a particular interval.
Properties of $\mathbb{R}$-linear Subspaces

- **Quantum Theory:** $\mathcal{H}$ Hilbert space. $U_g$ positive energy (anti)-unitary representation of the Möbius group.

- **Decompositions:** $U_g := T_I(x)\Lambda_I(y)P_I(z)$, and $J_I$
  
  - **Generators of $PSL(2,\mathbb{R})$:** Selfadjoint operators $H_I$, $D_I$ and $C_I$ satisfy:
    \[
    [H_I, D_I] = iH_I, \quad [C_I, D_I] = -iC_I, \quad [H_I, C_I] = 2iD_I.
    \]

  - $J_I$ the corresponding antiunitary transformation.

- **Remark:** A decomposition selects an interval $I$ in an abstract way. Thus intrinsically.
Properties of $\mathbb{R}$-linear Subspaces

Fix a particular decomposition, then

- Modular structure:
  - $\Delta_I := e^{-2\pi D_I}$ (modular operator)
  - $J_I$ (modular conjugation)

- Real subspaces from modular operators: $S_I := J_I \Delta_I^{1/2}$ and $\mathcal{K}_I := \{\psi | S_I \psi = \psi\}$

- From now on we choose the decomposition for the upper semicircle $I_1$
  (positive part of $\mathbb{P}\mathbb{R}$).
  $H, D, C$ the self adj. generators and $J$ the anti-unitary involution. $\Delta := \exp -2\pi D$
Digression: POVM

- **Pauli Theorem:** It is not possible to have a selfadjoint operator $X$, showing CCR with $P$ bounded from below.

- Gen. of rotation $(H + C)/2$ is positive, does not exists a self-adj. operator representing a global coordinate.

- Ordinary QM: $E$ energy and $T$ time. Usually this is circumvent enlarging the concept of observable to POVM. (Naimark).

- In KMS states $E$ is not bounded from below, then a selfadjoint $T$ operator exists. (Narnhofer, Thirring)

- We are searching for local coordinates for the interval $I$: it has to show CCR with the generator of modular transformation.
From positive inclusions: $[H, D] = iH \ [C, D] = -iC$

**Candidates** for $X$ showing CCR with $D$: $\log H$ and $\log C$

$$\gamma \log(C) - (1 - \gamma) \log H + f(D).$$

But we want it being compatible with emerging locality:

If $\psi \in \mathcal{K}_{[a,b] \subset I_1}$,

$$\log (a) \Vert \psi \Vert^2 \leq (\psi, X\psi) \leq \log (b) \Vert \psi \Vert^2.$$

**We have the following results**

- $D$ is positive on $\psi \in \mathcal{K}_{I_1}$.
  (See also Guido and Longo).
- For every $\psi \in \mathcal{K}_{[a,b] \subset I_1}$, the subsequent inequalities hold

$$a^2(\psi, H\psi) \leq (\psi, C\psi) \leq b^2(\psi, H\psi).$$
Some energy bounds

If $\psi \in \mathcal{K}_{I_1}$ then $(\psi, D\psi) \geq 0$.

**Proof steps:** $J \Delta^{1/2} \psi = \psi$ and $JDJ = -D$.

$F(\alpha) := (\psi, D\Delta^{\alpha} \psi)$, $F(0) = -F(1)$,

$\frac{d}{d\alpha} F(\alpha) \leq 0$ if $0 \leq \alpha \leq 1$. Then $F(0) \geq 0$.

For every $\psi \in \mathcal{K}_{[a,b] \subset I_1}$, the subsequent inequalities hold

$$a^2 (\psi, H\psi) \leq (\psi, C\psi) \leq b^2 (\psi, H\psi).$$

**Proof steps:** $U := e^{-iaH}$, $\psi \in \mathcal{K}_{[a,b]}$ then $\varphi := U\psi \in \mathcal{K}_{I_1}$

$$(\psi, C\psi) = (\varphi, C + 2aD + a^2 H\varphi) \geq (\varphi, 2aD + a^2 H\varphi) \geq (\psi, a^2 H\psi).$$
Modular coordinate

**Idea:** it seems possible to use “energies” for measuring positions. In fact, since log is a monotone function

\[ \log(a) \leq \left( \log \langle C \rangle_\psi - \log \langle H \rangle_\psi \right)/2 \leq \log(b), \]

where \( \langle C \rangle_\psi = (\psi, C\psi) \).

Eventually we shall see that

\[ X = \frac{1}{2} \log(H^{-1/2}CH^{-1/2}) \]

**NB** The domain needs to be fixed properly.
From $H, C, D$ generate a representation of $PSL(2, \mathbb{R})$ on $\mathcal{H}$.

- Decompose $\mathcal{H}$ in irreducible representations $\mathcal{H} = \bigoplus_i \mathcal{H}_i$.

\[
\tilde{H} := \frac{H^2}{2}, \quad \tilde{D} := \frac{D}{2}, \quad \tilde{C} := \frac{H^{-1/2}CH^{-1/2}}{2}
\]

- Enjoy $sl(2, \mathbb{R})$ commutation relations.
- There is a dense set of analytic vectors on every $\mathcal{H}_i$.
- Generate a positive-energy unitary representation $\tilde{U}$ of the covering group of $SL(2, \mathbb{R})$ on $\mathcal{H}$.
- For the lowest eigenvalues of rotation gen. we have $\tilde{k} = k/2 + 1/4$

Let $\psi \in \mathcal{K}_I$ where $I = [a, b] \subset I_1$ then

\[
\frac{a^2}{2} \|\psi\|^2 < (\psi, \tilde{C}\psi) < \frac{b^2}{2} \|\psi\|^2.
\]
Position Operator

Since the logarithm is also an operator monotone function, we get
\[ X := \frac{1}{2} \log(2\tilde{C}). \]

- It is self-adjoint on a suitable domain.
- It shows CCR with \( D \):
  \[ [D, X] := i \]
- It is compatible with emerging locality: \( \psi \in \mathcal{K}[a,b] \subset l_1 \)
  \[ \log(a) \|\psi\|^2 \leq (\psi, X\psi) \leq \log(b) \|\psi\|^2 \]
Massless scalar field on $\mathbb{R}_{1,1}$: coordinate of a Wedge

- 2D Minkowski: $ds^2 = -dt^2 + dx^2$,
- Massless KG equation has two modes, \textit{in-} and \textit{out-}
- One-particle Hilbert space is $L(\mathbb{R}^+, dE) \oplus L(\mathbb{R}^+, dE)$.
- On $L(\mathbb{R}^+, dE)$, the representation of the Möbius group is generated by:

\[
H := E, \quad D = -i\sqrt{E} \frac{d}{dE} \sqrt{E}, \quad C = -\sqrt{E} \frac{d^2}{dE^2} \sqrt{E},
\]

and the anti-unitary involution: the complex conjugation.
If we read them in the following coordinates: $\mathbb{R}_{1,1} := -dv \, du$

The action of $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ on wave-function $\partial_v \psi(v)$ reads:

$$U_g \partial_v \psi(v) = \frac{1}{(cv' + d)^2} \partial_{v'} \psi(v'), \quad v' = \frac{dv - b}{a - cv}$$

- Emerging localization is compatible with that of the wedges.
- A Model for Quantum coordinates inside a wedge.
- The scheme, does not work for massive fields: the one particle Hilbert space is only one $L^2(\mathbb{R}^+, dh)$. ($h = p + \sqrt{p^2 + m^2}$)
- In this case we get at most an operator measuring a spatial coordinate.
Summary

- Localization can arise from the group properties.

- Also in the case of Möbius covariant theory. \textit{(Positive energy representation)}

- An operator representing a local coordinate arises modifying the energy and the conformal energy
  - CCR with generator of modular transformation.
  - expectation values on local wavefunction compatible with localization.